# ON THE APPROXIMATE SOLUTION OF SOME INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY AND MATHEMATICAL PHYSICS

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In this paper the development of the method presented in [1] is carried out with application to two types of integral equations encountered in mathematical physics in the investigation of many mixed problems with circular separation line of boundary conditions and in the investigation of plane mixed problems.

The algorithm is given for reducing these integral equations to solution of equivalent infinite linear algebraic systems. It is proved that the resulting infinite systems are quasi completely regular for sufficiently large values of dimensionless parameter $\lambda$ which enters into the systems. It is shown that reduction (truncation) of infinite systems results in finite systems of linear algebraic equations with almost triangular matrices. The last circumstance simplifies considerably the solution of these finite systems after which the solution of initial integral equations is found from simple equations. For given accuracy of the approximate solution and decrease of parameter $\lambda$ the number of equations in reduced systems increases.

As an example the solution is presented for the axisymmetric problem of a die acting on an elastic layer lying without friction on a rigid foundation.

**1. Basic integral equation of mixed problems with circular separation line of boundary conditions; general form of solution of this equation.** The following integral Eq. is examined

$$\int_0^1 \varphi(\rho)\rho d\rho \int_0^\infty L(u) J_n\left(\frac{u\rho}{\lambda}\right) J_n\left(\frac{ur}{\lambda}\right) du = \lambda f(r) \qquad (0 \leqslant r \leqslant 1) \tag{1.1}$$

Here $J_n(x)$ is the Bessel function. Let us assume that function $L(u)$ satisfies the following properties:

$$L(u) = 1 + O(e^{-\nu u}) \quad \text{for} \quad u \to \infty \quad (\nu > 0), \quad L(u) = O(u) \quad \text{for} \quad u \to 0 \tag{1.2}$$

for all $u \in (0, \infty)$ the function $L(u)$ is continuous together with all derivatives.

To find solutions of integral Eq. (1.1), it is sufficient to learn how to solve the simpler integral Eq.

$$\int_0^1 \psi(\rho)\rho d\rho \int_0^\infty L(u) J_0\left(\frac{u\rho}{\lambda}\right) J_0\left(\frac{ur}{\lambda}\right) du = \lambda g(r) \qquad (0 \leqslant r \leqslant 1) \tag{1.3}$$

In fact we can prove that if the general solution of differential Eq.

$$r^n A^n(g) = f(r) \qquad \left(A = \frac{1}{r}\frac{d}{dr}\right) \tag{1.4}$$

which is determined with accuracy to $n$ arbitrary constants, is taken as function $g(r)$ in Eq. (1.3) and if subsequently the solution $\psi(\rho)$, which for $\rho = 1$ becomes zero together with all its derivatives to the order of $(n-1)$ included, is determined, for the integral equation (1.3) then the solution $\varphi(\rho)$ of the integral Eq. (1.1) is determined from Eq.

$$\varphi(\rho) = \rho^n A^n(\psi) \tag{1.5}$$

We note that all arbitrary constants entering into the function $g(r)$ found from Eq. (1.4) are determined from the following conditions when the indicated algorithm is satisfied:

$$\psi^{(k)}(1) = 0 \qquad (k = 0, 1, \ldots, n-1) \tag{1.6}$$

On the basis of what was stated, in this manner, everything following will be devoted only to a study of integral Eq. (1.3). Utilizing integral [2]

$$\int_0^\infty J_0(\mu u) J_0(\nu u)\, du = \frac{2}{\pi(\mu+\nu)} K\left(\frac{2\sqrt{\mu\nu}}{\mu+\nu}\right) \tag{1.7}$$

we rewrite the integral Eq. (1.3) in the form

$$\int_0^1 \psi(\rho) K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)\frac{\rho\, d\rho}{r+\rho} = \frac{\pi}{2} g(r) + \int_0^1 \psi(\rho) F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right)\rho\, d\rho \qquad (0 \leqslant r \leqslant 1)$$

$$F(\mu, \nu) = \frac{\pi}{2\lambda}\int_0^\infty [1 - L(u)] J_0(\mu u) J_0(\nu u)\, du \tag{1.8}$$

Here and in (1.7) $K(x)$ is the complete elliptic integral of the first kind. Based on properties (1.2) of function $L(u)$ it is easy to show that the even function $F(\mu, \nu)$ is continuous together with all its derivatives with respect to all variables in the square $-\infty < 0 \leqslant (\mu, \nu) < \infty$.

We shall seek the solution $\psi(\rho)$ of Eq. (1.8) in the class $L(S_1)$ of absolutely summable functions in the circle $S_1(\rho \leqslant 1)$, then for $\lambda \to \infty$ the integral Eq. (1.8) degenerates into the following:

$$\int_0^1 \psi_0(\rho) K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)\frac{\rho\, d\rho}{r+\rho} = \frac{\pi}{2} g(r) \qquad (0 \leqslant r \leqslant 1) \tag{1.9}$$

As is known, the axisymmetric contact problem of an elastic half-space is reduced to such an equation.

Many authors found the solution of integral Eq. (1.9) in closed form by different methods.

Here in our opinion the simplest method of solving integral Eq. (1.9) will be shown, and subsequently an investigation of the structure and differential properties of function $\psi_0(\rho)$ will be carried out.

Utilizing (1.7), the integral Eq. (1.9) is reduced to its equivalent conjugate integral Eq.

$$\int_0^\infty \Psi_0(u) J_0(ur)\, u\, du = 0 \quad (r > 1); \qquad \int_0^\infty \Psi_0(u) J_0(ur)\, du = g(r) \quad (r \leqslant 1) \tag{1.10}$$

where

$$\psi_0(r) = \int_0^\infty \Psi_0(u) J_0(ur)\, u\, du \qquad \text{for } r \leqslant 1 \tag{1.11}$$

The first relationship (1.10) is multiplied by $r(r^2 - t^2)^{-1/2} dr$ and integrated with respect to $r$ from $t$ to $\infty$, the second relationship (1.10) is multiplied term-by-term by $r(t^2 - r^2)^{-1/2} dr$ and integrated with respect to $r$ from 0 to $t$. Then having utilized Eqs.

$$\int_0^t \frac{rJ_0(r\gamma)}{\sqrt{t^2-r^2}}\,dr=\frac{\sin\gamma t}{\gamma},\qquad \int_t^\infty \frac{rJ_0(r\gamma)}{\sqrt{r^2-t^2}}\,dr=\frac{\cos\gamma t}{\gamma} \tag{1.12}$$

the following conjugate integral Eq. is obtained:

$$\int_0^\infty \Psi_0(\gamma)\cos\gamma t\,d\gamma=0 \quad (t>1);\qquad \int_0^\infty \Psi_0(\gamma)\sin\gamma t\frac{d\gamma}{\gamma}=\int_0^t\frac{g(r)\,r\,dr}{\sqrt{t^2-r^2}} \quad (t\leqslant 1) \tag{1.13}$$

Now the function $\psi^*(t)$ is introduced into the examination. This function is connected to $\Psi_0(u)$ through relationships

$$\int_0^\infty \Psi_0(u)\cos ut\,du=\begin{cases}0 & (t>1)\\ \psi^*(t) & (t\leqslant 1)\end{cases},\qquad \Psi_0(u)=\frac{2}{\pi}\int_0^1\psi^*(\tau)\cos u\tau\,d\tau \tag{1.14}$$

Differentiating the second Eq. (1.13) term-by-term with respect to $t$, we find

$$\psi^*(t)=\frac{d}{dt}\int_0^t\frac{g(r)\,r\,dr}{\sqrt{t^2-r^2}} \tag{1.15}$$

We shall now establish the relationship between $\psi_0(r)$ and $\psi^*(t)$. Substituting Expression $\Psi_0(u)$ in the form (1.14) into (1.11) and utilizing the integral [2]

$$\int_0^\infty \sin u\tau J_0(ur)\,du=\begin{cases}0 & (0<\tau<r)\\ (\tau^2-r^2)^{-1/2} & (0<r<\tau)\end{cases} \tag{1.16}$$

we obtain

$$\psi_0(r)=-\frac{2}{\pi}\frac{d}{dr}r\int_r^1\frac{\psi^*(\tau)\,d\tau}{\tau\sqrt{\tau^2-r^2}} \tag{1.17}$$

Thus Eqs. (1.15) and (1.17) give the solution of the integral Eq. (1.9). We can show, but we shall not dwell on it here in detail, that this solution has a meaning(*) at least for $g(r)\in H^\alpha(S_1)$, $\alpha>1/2$.

In the following it will be assumed that $g''(r)$ is bounded when $r\in[S_1]$. In this case we obtain from (1.15) without difficulty

$$\psi^{*\prime}(t)=\int_0^t\frac{g'(r)\,dr}{\sqrt{t^2-r^2}}+\int_0^t\frac{rg''(r)\,dr}{\sqrt{t^2-r^2}} \tag{1.18}$$

Theorem 1. If $g''(r)$ is bounded, then $\psi^{*\prime}(t)\in H^{1/2}(S_1)$. From the boundedness of $g''(r)$ in the circle $S_1$ it follows that $|g'(r)|\leqslant Cr$ for all $r\in S_1$. Then on the basis of (1.18)

$$|\psi^{*\prime}(t)|\leqslant(C+C')\int_0^t\frac{r\,dr}{\sqrt{t^2-r^2}}=Dt\quad(t\in S_1) \tag{1.19}$$

It remains to be shown that

$$\psi^{*\prime}(t)\in H^{1/2}(S_{1-\varepsilon}) \tag{1.20}$$

where $S_{1-\varepsilon}$ is the circle with unit radius with $\varepsilon$-region of point $t=0$ excluded. We note that condition (1.20) will be fulfilled if the stronger statement $\psi^{*\prime}(t)\in H^{1/2}(\varepsilon,1)$ is proved, i.e.

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*) This indicates that $|f(P)-f(Q)|\leqslant AR^\alpha_{PQ}$ ($R_{PQ}$ is the distance between points $P$ and $Q$) for any $P$ and $Q\in S_1$.

$$|\psi^{*\prime}(t)-\psi^{*\prime}(\tau)|\leqslant D\,|t-\tau|^{1/2} \tag{1.21}$$

for all $t$ and $\tau\in[\varepsilon,1]$.

For the first integral in (1.18) the statement made is obvious. Let us examine the second integral in more detail. We estimate the modulus of difference ($t>\tau$).

$$\left|\int_0^t \frac{rg''(r)\,dr}{\sqrt{t^2-r^2}}-\int_0^\tau \frac{rg''(r)\,dr}{\sqrt{\tau^2-r^2}}\right|\leqslant\left|\int_0^\tau rg''(r)\left(\frac{1}{\sqrt{t^2-r^2}}-\frac{1}{\sqrt{\tau^2-r^2}}\right)dr\right|+\left|\int_\tau^t \frac{rg''(r)\,dr}{\sqrt{t^2-r^2}}\right| \tag{1.22}$$

Now we estimate the first integral of the right part of (1.22). Utilizing the obvious identity

$$(\tau^2-r^2)^{-1/2}=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}(t^2-\tau^2)^n(t^2-r^2)^{-(n+1/2)} \tag{1.23}$$

we rewrite it in the form

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}(t^2-\tau^2)^n\left|\int_0^\tau\frac{rg''(r)\,dr}{(t^2-r^2)^{n+1/2}}\right|\leqslant B\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}(t^2-\tau^2)^n\int_0^\tau\frac{r\,dr}{(t^2-r^2)^{n+1/2}}=$$

$$=B\sqrt{t^2-\tau^2}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left[1-\left(1-\frac{\tau^2}{t^2}\right)^{n-1/2}\right]\leqslant$$

$$\leqslant\sqrt{2}\,B\,|t-\tau|^{1/2}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(2n)!!}=\sqrt{2}\,B\,|t-\tau|^{1/2} \tag{1.24}$$

The second integral of the right of (1.19) is estimated without difficulty

$$\left|\int_\tau^t\frac{rg''(r)\,dr}{\sqrt{t^2-r^2}}\right|\leqslant B\sqrt{t^2-\tau^2}\leqslant\sqrt{2}\,B\,|t-\tau|^{1/2} \tag{1.25}$$

On the basis of previous proofs we rewrite Eq. (1.17) in the form

$$\psi_0(r)=\frac{2}{\pi}\left[\frac{\psi^*(1)}{\sqrt{1-r^2}}-\int_r^1\frac{\psi^{*\prime}(\tau)\,d\tau}{\sqrt{\tau^2-r^2}}\right] \tag{1.26}$$

Theorem 2. If $g''(r)$ is bounded, then $\psi_0(r)$ has the form

$$\psi_0(r)=\omega(r)(1-r^2)^{-1/2} \tag{1.27}$$

where $\omega(r)\in H^{1/2}(S_1)$, i.e. $\psi_0(r)\in L(S_1)$.

The procedure of proof is analogous to Theorem 1. At first it is shown that $|\omega(r)-\omega(0)|\leqslant Er$ and then $\omega(r)\in H^{1/2}(\varepsilon,1)$. Let us formulate another more general theorem.

Theorem 3. If function $g(r)$ is such that its $n+2$ derivative is bounded for $r\in|S_1|$ then function $\psi_0(r)$ has the form (1.27), where $\omega^{(n)}(r)\in H^{1/2}(S_1)$. The proof is carried out analogously to what was presented above.

We shall now seek the solution of Eq. (1.8) from the class $L(S_1)$ in the form

$$\psi(\rho)=\psi_0(\rho)+\psi_1(\rho) \tag{1.28}$$

where $\psi_0(\rho)$ is the solution of integral Eq. (1.9) determined by relationships (1.15) and (1.17). For the correction function $\psi_1(\rho)$ we obtain the integral Eq.

$$\int_0^1\psi_1(\rho)K\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)\frac{\rho\,d\rho}{r+\rho}=\frac{\pi}{2}g_*(r)+\int_0^1\psi_1(\rho)F\left(\frac{r}{\lambda},\frac{\rho}{\lambda}\right)\rho\,d\rho \qquad (0\leqslant r\leqslant 1)$$

$$g_*(r) = \frac{2}{\pi}\int_0^1 \psi_0(\rho) F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right)\rho\, d\rho \qquad (1.29)$$

We note that by virtue of properties of function $F(\mu, \nu)$ and condition $\psi(\rho) \in L(S_1)$, or by taking into account the Theorem 2 $\psi_1(\rho) \in L(S_1)$, the entire right side of the integral Eq. (1.29), as function of $r \in [S_1]$ is continuous with all derivatives.

Then on the basis of Theorem 3 we can conclude that for any value $\lambda \in (0, \infty)$ the general solution of the integral Eq. (1.29), if it exists in $L(S_1)$, has the form

$$\psi_1(r) = \Omega(r)(1 - r^2)^{-1/2} \qquad (1.30)$$

where $\Omega(r)$ with all the derivatives is a continuous function for $r \in [S_1]$.

Thus, to find the general solution of the integral Eq. $(S_1)$, it is necessary to find function $\Omega(r)$. Section 2 will be devoted to this subject.

**2. Reduction of integral Eq. (1.29) to solution of an infinite system of linear algebraic equations.** Let us represent the function $F(\mu, \nu)$ of the type (1.8) in the form of the following double series with respect to even Legendre polynomials

$$F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e_{ij}(\lambda) P_{2i}(\sqrt{1-r^2}) P_{2i}(\sqrt{1-\rho^2}) \qquad (2.1)$$

Functions $g_*(r)$ and $\Omega(r)$ which enter into Eqs. (1.29) and (1.30) are also expanded in series (*)

$$g_*(r) = \sum_{i=0}^{\infty} R_{i*} P_{2i}(\sqrt{1-r^2}), \qquad \Omega(r) = \sum_{i=0}^{\infty} S_i P_{2i}(\sqrt{1-r^2}) \qquad (2.2)$$

By virtue of properties of functions $F(\mu, \nu)$, $g_*(r)$ and $\Omega(r)$ pointed out in Section 1, series (2.1) and (2.2) converge uniformly to $F(\mu, \nu)$ with respect to all variables $(r, \rho) \in [0, 1]$ and arbitrary values of parameter $\lambda \in (0, \infty)$ and to $g_*(r)$ and $\Omega(r)$ for all $r \in S_1$, respectively.

Utilizing the known orthogonality property of Legendre polynomials [2]

$$\int_0^1 P_{2i}(\sqrt{1-x^2}) P_{2j}(\sqrt{1-x^2}) \frac{x\,dx}{\sqrt{1-x^2}} = \begin{cases} 0 & (i \neq j) \\ (4i+1)^{-1} & (i = j) \end{cases} \qquad (2.3)$$

we obtain the following expression for coefficients $e_{ij}(\lambda)$ of series (2.1) (2.4)

$$e_{mn} = (4m+1)(4n+1)\int_0^1\int_0^1 F\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) P_{2m}(\sqrt{1-x^2}) P_{2n}(\sqrt{1-y^2}) \frac{xy\,dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$$

Now substituting into (2.4) Expression $F(\mu, \nu)$ of the form (1.8) and utilizing the integral [2]

$$\int_0^1 J_0(bx) P_{2i}(\sqrt{1-x^2}) \frac{x\,dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2}}\frac{(2i-1)!!}{(2i)!!}\frac{1}{\sqrt{b}} J_{2i+1/2}(b) \qquad (2.5)$$

we obtain the other Eq. for $e_{mn}(\lambda)$: (2.6)

$$e_{mn} = (4m+1)(4n+1)\frac{\pi^2 (2m-1)!!\,(2n-1)!!}{4\,(2m)!!\,(2n)!!}\int_0^\infty [1 - L(u)] J_{2m+1/2}\left(\frac{u}{\lambda}\right) J_{2n+1/2}\left(\frac{u}{\lambda}\right)\frac{du}{u}$$

*) We note that Legendre polynomials for finding approximate solutions of integral equations similar to (1.3) were also utilized in paper [3]. The main advantage of the approach presented in this Section is in our opinion the representation of $F(\mu, \nu)$ in the form (2.1)

We shall proceed to the detemination of coefficients $R_{i*}$. Utilizing the second Eq. (1.29) and (2.1) we find

$$R_{i*}=\sum_{n=0}^{\infty} e_{in}(\lambda)\,\Psi_n, \qquad \Psi_n=\frac{2}{\pi}\int_0^1 \psi_0(\rho)\,P_{2n}(\sqrt{1-\rho^2})\,\rho\,d\rho \tag{2.7}$$

The integral (2.7) is estimated using the following artificial step. Both parts of integral Eq. (1.9) are multiplied term-by-term by

$$r\,(1-r^2)^{-1/2}\,P_{2n}(\sqrt{1-r^2})\,dr$$

and integration is performed with respect to $r$ from zero to one. Then rearranging integrals in the left part of the obtained relationship and utilizing Eq. [3]

$$\int_0^1 \frac{\tau P_{2m}(\sqrt{1-\tau^2})}{\sqrt{1-\tau^2}}\,K\left(\frac{2\sqrt{\tau t}}{\tau+t}\right)\frac{d\tau}{\tau+t}=\frac{\pi^2\,[(2m-1)!!]^2}{4\,[(2m)!!]^2}\,P_{2m}(\sqrt{1-t^2}) \tag{2.8}$$

we obtain for integrals $\Psi_n$ following Expressions

$$\Psi_n=\frac{4\,[(2n)!!]^2}{\pi^2\,[(2n-1)!!]^2}\,\frac{R_n}{(4n+1)}, \quad R_n=(4n+1)\int_0^1 g(r)\,P_{2n}(\sqrt{1-r^2})\,\frac{r\,dr}{\sqrt{1-r^2}} \tag{2.9}$$

We note that quantities $R_n$ are coefficients of the expansion of function $g(r)$ into a series with respect to Legendre polynomials of the form (2.2).

Finally we obtain relationships for determination of coefficients $S_i$ in the second Eq. (2.2). Substituting functions $\psi_1(\rho)$, $g_*(r)$ and $F(\mu,\nu)$ in the form (1.30), (2.2) and (2.1) into the integral Eq. (1.29) and computing integrals from Eqs. (2.3) and (2.8) we obtain a relationship which in its left and right part contains series in Legendre polynomials. Equating coefficients of both parts for polynomials of the same number, we obtain an infinite system of linear algebraic equations for determination of $S_i$

$$S_i\,\frac{\pi}{2}\,\frac{[(2i-1)!!]^2}{[(2i)!!]^2}=R_{i*}+\frac{2}{\pi}\sum_{k=0}^{\infty} S_k\,\frac{e_{ik}(\lambda)}{4k+1} \qquad (i=0,1,\ldots,\infty) \tag{2.10}$$

**3. Analysis of infinite system (2.10).** Let us rewrite system (2.10) in a more convenient form

$$x_i=\sum_{k=0}^{\infty} a_{ik}x_k+R_{i*} \qquad (i=0,1,\ldots,\infty) \tag{3.1}$$

Here

$$x_i=\frac{\pi}{2}\,\frac{[(2i-1)!!]^2}{[(2i)!!]^2}\,S_i, \qquad a_{mn}=\frac{4}{\pi^2}\,\frac{[(2n)!!]^2}{[(2n-1)!!]^2}\,\frac{e_{mn}(\lambda)}{(4n+1)}=$$

$$=(4m+1)\,\frac{(2n)!!\,(2m-1)!!}{(2m)!!\,(2n-1)!!}\int_0^\infty [1-L(u)]\,J_{2m+1/2}\left(\frac{u}{\lambda}\right)J_{2n+1/2}\left(\frac{u}{\lambda}\right)\frac{du}{u} \tag{3.2}$$

Now let us find asymptotic equations for coefficients $a_{mn}$ of the form (3.2) for large and small values of parameter $\lambda$.

Making use of a known representation $J_\nu(x)$ in the form of a series in powers of $x$ we obtain for $a_{mn}$ when $\lambda\to\infty$ the following Expression:

$$a_{mn}=\frac{2}{\pi\lambda^{2p+1}}\,\frac{(2n)!!\,(2m-1)!!}{(2m)!!\,(2n-1)!!\,(4m-1)!!\,(4n+1)!!}\Big[I_p-\frac{2p+3}{(4m+3)(4n+3)\lambda^2}\,I_{p+1}+$$

$$+\frac{4(p^2-mn)+13p+{}^{55}/_4}{(4m+3)(4m+5)(4n+3)(4n+5)\lambda^4}\,I_{p+2}+O(\lambda^{-6})\Big] \qquad (p=m+n) \tag{3.3}$$

$$I_k = \int_0^\infty [1 - L(u)]\, u^{2k}\, du$$

Eq. (3.3) is simplified assuming that $m$ and $n$ are large; for this at first we find asymptotics of integrals $I_k$ of the form (3.3) for large $k$.

Taking into consideration the first of relationships (1.2) we shall have for $k \to \infty$

$$I_k = O\,[(2k)!\, \nu^{-(2k+1)}] \tag{3.4}$$

Now using Stirling's formula we obtain the following asymptotic expression for $a_{mn}$ for large $m$, $n$ and $\lambda$:

$$a_{mn} = O\left[\frac{p^{2p+1/2}}{(2\lambda\nu)^{2p+1}\, m^{2m+1/2} n^{2n+1/2}}\right] \tag{3.5}$$

In order to investigate the behavior of coefficients $a_{mn}$ for small $\lambda$, a substitution of the variable is carried out under the integral in (3.2), writing $u/\lambda = \alpha$. Then utilizing the second property (1.2) of function $L(u)$ and returning to the old variable $u$ we shall have for $\lambda \to 0$

$$a_{mn} = (4m+1)\frac{(2n)!!\,(2m-1)!!}{(2m)!!\,(2n-1)!!}\int_0^\infty J_{2m+1/2}\left(\frac{u}{\lambda}\right) J_{2n+1/2}\left(\frac{u}{\lambda}\right)\frac{du}{u} \tag{3.6}$$

Now computing the integral [2], we finally obtain

$$a_{mn} = 0\,(m \neq n),\quad a_{mn} = 1 \tag{3.7}$$

We shall prove that for large values of parameter $\lambda$ the infinite system (3.1) is quasi-completely regular. For this purpose let us examine the series

$$b_i = \sum_{k=0}^{\infty} |a_{ik}| \qquad (i = 0, 1, \ldots, \infty) \tag{3.8}$$

Making use of the estimate (3.5) it is not difficult to prove that series (3.8) converge for any large but finite $i$, if the parameter $\nu > (2\nu)^{-1}$.

Now let us clarify the behavior of sums $b_i$ for $i \to \infty$. Let us examine Expression

$$b_i^* = \frac{1}{i}\sum_{k=0}^{i} i\,|a_{ik}| \tag{3.9}$$

It is clear that $b_i^* \to b_\infty$ for $i \to \infty$; in addition, we can state on the basis of the theorem of Stolz [4] that

$$\lim_{i\to\infty} b_i^* = \lim_{i\to\infty} i\,|a_{ii}| \tag{3.10}$$

From here, taking into account the estimate (3.5) it follows without difficulty that $b_\infty = 0$ if $\lambda > \nu^{-1}$.

In this manner sums $b_i$ approach 0 when $i \to \infty$, then starting from same $i = i_0$ we shall have $b_i < 1 - \varepsilon$, which indicates quasi-complete regularity of the system (3.1) for all $\lambda > \nu^{-1}$. In addition to this it is clear that free terms $R_{i*}$ of system (3.1) are bounded from above and for $i \to \infty$ approach zero by virtue of uniform convergence of first series (2.2).

For small values of parameter $\lambda$ the infinite system (3.1) becomes unstable because according to the estimate (3.7) its determinant approaches zero.

For finding approximate solutions of the infinite system (3.1) it is convenient to take advantage of the method of reduction (truncation). In this case the following finite system is obtained

$$x_i^n = \sum_{k=0}^{n-i} a_{ik} x_k^n + R_{i*}^n \qquad (i = 0, 1, \ldots, n) \tag{3.11}$$

The superscript $n$ with quantities $x_i$ indicates that solution is carried out for a system $n+1$ linear algebraic equations obtained by reduction (truncation) of an infinite system. The same index with quantity $R_{i*}$ indicates that in Eq. (2.7) which determines these coefficients it is necessary to carry out the summation with respect to $k$ up to $n$ and not up to $\infty$.

The solution of the system of linear algebraic Eqs. (3.11) is carried out sufficiently simply for any $n$ due to the fact that their coefficients form an almost triangular matrix. After determination of quantities $x_i^n$ from system (3.11) the approximate solution of integral Eq. (1.3) is found from Eqs. (1.28), (1.17), (1.15) and (1.30), the second Eq. (2.2) and the first Eq. (3.2). In this case in (2.2) the summation with respect to $i$ is carried out to $n$ and not to $\infty$.

We note that for a given accuracy of the approximate solution of the integral Eq. (1.3) and decrease in parameter $\lambda$ it is necessary to increase the quantity of $n+1$ equations in the reduced (truncated) system (3.11). In fact for $\lambda = 0$ we shall have on the basis of Eqs. (3.7) and the second Eq. (3.2) for coefficients $e_{mn}(\lambda)$ of series (2.1)

$$e_{mn} = 0 \quad (m \neq n), \qquad e_{mm} = \frac{\pi^2 (4m+1) [(2m-1)!!]^2}{4 [(2m)!!]^2} \tag{3.12}$$

or applying Stirling's formula, we find for large $m$, $e_{mm} = \pi$. Now, making use of this last fact and the asymptotics of Legendre polynomials [2] for large $m$, it is easy to show that the series (2.1) for function $F(\mu, \nu)$ diverges for $\lambda = 0$ on the line $\mu = \nu$. Consequently, on decreasing $\lambda$ the convergence on this line will become poorer and for preservation of given accuracy of the approximate solution the number of equations in system (3.11) must be increased.

In this manner good convergence of the method presented above for approximate solution of integral Eq. (1.3) must be expected only for large and intermediate values of parameter $\lambda$.

For final clarification of limits of rational utilization of the presented method we briefly touch on other methods of approximate solution of integral Eq. (1.3).

By the method of large $\lambda$ (see [5 and 6]) we can obtain the following approximate solution of (1.3): (3.13)

$$\psi(r) = -\frac{2}{\pi}\frac{d}{dr} r \int_r^1 \frac{\psi^*(\tau)\, d\tau}{\tau\sqrt{\tau^2 - r^2}} + \frac{2}{\pi\sqrt{1-r^2}} \int_0^1 \psi^*(\tau) \Big[ q\Big(1 + q + q^2 + q^3 - \frac{2}{3}p\Big) -$$

$$- p(2r^2 + \tau^2 - 1)(1+q) \Big] d\tau + O(\lambda^{-6}) \qquad \Big(q = \frac{2}{\pi\lambda} I_0,\ p = \frac{1}{\pi\lambda^3} I_1\Big)$$

where $I_0$ and $I_1$ have the form (3.3) while the function $\psi^*(\tau)$ is determined by Expression (1.15).

The zeroth term of the asymptotic for the solution of the integral Eq. (1.3) can be found for small $\lambda$ in the following manner.

Let us rewrite the integral Eq. (1.3) utilizing the asymptotic representation of the Bessel function for large values of the argument (small $\lambda$) in the form

$$\int_{-1}^{1} \chi(\rho) M\Big(\frac{\rho - r}{\lambda}\Big) d\rho = \pi h(r) \qquad (|r| \leqslant 1) \tag{3.14}$$

$$M(x) = \int_0^\infty \frac{L(u)}{u} \cos ux\, du, \qquad \chi(\rho) = \psi(|\rho|)\sqrt{|\rho|}, \qquad h(r) = g(|r|)\sqrt{|r|}$$

Now let us change to new variables in the integral Eq. (3.14) according to following Formulas [7 and 8] (*):

$$\tau = \frac{h(1) - h(\rho)}{\lambda h'(1)}, \qquad t = \frac{h(1) - h(r)}{\lambda h'(1)} \tag{3.15}$$

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*) It is assumed here that function $h(r)$ is strictly monotonous with respect to $r$ for $0 < |r| \leqslant 1$. This limitation is not significant because $h(r)$ can always be represented in the form of a sum of two strictly monotonous functions.

Back substitutions for small $\lambda$ permit the representation of $|\rho|$ and $|r|$ in a unique manner through asymptotic expressions $|\rho| = 1 - \lambda\tau + \ldots$ and $|r| = 1 - \lambda t + \ldots$ Therefore we shall have

$$\int_0^{c/\lambda} \chi^*(\tau) M\left(\frac{2}{\lambda}\right) d\tau + \int_0^{c/\lambda} \chi^*(\tau) M(\tau - t)\, d\tau = \frac{\pi}{\lambda}[h(1) - \lambda t h'(1)] \qquad \left(0 \leqslant t \leqslant \frac{c}{\lambda}\right)$$

$$c = [h(1) - h(0)][h'(1)]^{-1}, \qquad \chi^*(\tau) \equiv \chi(\rho) \tag{3.16}$$

If it is now taken into consideration that on the basis of second Eq. (1.2) the kernel $M(x) \sim \delta(x)$ for $|x| \to \infty$ ($\delta(x)$ is Dirac's delta function) and if in the left part of the integral Eq. (3.16) the parameter $\lambda$ is allowed to approach zero, then the determination of the zeroth term of the asymptotic of the solution of the integral Eq. (1.3) for small $\lambda$ is reduced to the determination of the solution for the following integral equation of Wiener-Hopf

$$\int_0^{\infty} \chi^*(\tau) M(\tau - t)\, d\tau = \frac{\pi}{\lambda}[h(1) - \lambda t h'(1)] \qquad (0 \leqslant t < \infty) \tag{3.17}$$

Construction of the zeroth term of the asymptotic of the solution for small $\lambda$ by the method presented in [9] can also be applied to the solution of the integral Eq. (3.17), however, with a different right-hand part. We shall not dwell on this in detail.

Examination of concrete problems shows that for large $\lambda$ the asymptotic solution of the form (3.13) and the zeroth term of the asymptotic of the solution for small $\lambda$ give as a rule reliable results for $2 \leqslant \lambda < \infty$ and $0 < \lambda \leqslant ½$, respectively.

In the paper [10] the possibility to find the complete asymptotic for the solution of the integral Eq. (1.3) is shown for small $\lambda$ under the assumption that the function $L(u)$ is meromorphic. An analysis of this complete asymptotic in the examination of concrete problems apparently would permit to make the matching with the asymptotic solution (3.13) for large $\lambda$. However, the practical construction of the complete asymptotic of the solution for small $\lambda$ and its subsequent numerical analysis present significant difficulties.

In this manner in accordance with everything stated above, the method presented in this paper for an approximate solution of integral Eq. (1.3) in our opinion must basically serve as the "connecting bridge" between the asymptotic solution for large $\lambda$ of the form (3.13) and the zeroth member of the asymptotic of the solution for small $\lambda$.

**4. Example.** Let us examine the axisymmetric problem of action of a rigid die on an elastic layer which is lying without friction on a rigid foundation. Friction forces are assumed to be absent between the die and the layer. Utilizing the integral transformation of Hankel we can reduce the contact problem under examination to a solution of an integral equation which in nondimensional coordinates has the form (1.3) $\psi(\rho)$ is the unknown pressure between the die and the layer on the line of contact, $\lambda = h/a$, $g(r) = a^{-1}\Delta\gamma(r)$, $h$ is the thickness of the layer, $a$ is the radius of the contact region, $\Delta = G(1-\sigma)^{-1}$, $G$ and $\sigma$ are elasticity constants of the layer and $\gamma(r)$ is the settling of points of the boundary of the layer under the die. The function $L(u)$ can be represented in the form

$$L(u) = \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u} \tag{4.1}$$

It is easy to show that function $L(u)$ of the form (4.1) satisfies conditions (1.2); $\nu = 2$ in this case. It follows from this that for the problem under consideration the infinite system of linear algebraic Eqs. (3.1) will be quasi-completely regular for all $\lambda > ½$. The values of constants $a_{mn}(\lambda)$ entering into system (3.1) can be found for large values of parameter $\lambda$ ($\lambda \geqslant 2$) from the asymptotic Eq. (3.3). For other values of $\lambda$ they can be determined by methods of numerical integration from Eqs. (2.4) and (2.6) and the second Eq. (3.2). In this manner for $\lambda = 1$ the following values of constants $a_{mn}(\lambda)$ are obtained:

$$a_{00} = 0.3447,\ a_{10} = 0.1678,\ a_{01} = 0.008389,\ a_{20} = 0.01313,\ a_{11} = 0.02367,\ a_{02} = 0.0002052$$

We shall not dwell on the technique of calculations. It is only noted that a significant simplification of the computational algorithm was achieved by utilizing tables of the function [6]

$$F(k) = \int_0^\infty [1 - L(u)]\, J_0(uk)\, du \tag{4.2}$$

(here $L(u)$ has the form (4.1), and $J_0(x)$ is Bessel function) and relationships

$$F(\mu, \nu) = \frac{1}{2\pi} \int_0^{2\pi} F(R)\, d\varphi \qquad (R = \sqrt{\mu^2 + \nu^2 - 2\mu\nu \cos\varphi}) \tag{4.3}$$

By the method presented in this paper it now is not difficult to obtain an approximate solution of the contact problem under examination for $\lambda = 1$ for a die with one or another base. For example, if $\gamma(r) \equiv \gamma$ (plane die), the approximate solution corresponding to the case $n = 1$ in (3.11) has the form

$$\psi(r) = \Delta\gamma/a\ (1 - r^2)^{-1/2}\ (1.7680 - 0.5532\ r^2) \tag{4.4}$$

increasing $n$ by one we obtain

$$\psi(r) = \Delta\ \gamma/a\ (1 - r^2)^{-1/2}\ (1.8116 - 0.7190\ r^2 + 0.1263 r^4) \tag{4.5}$$

Further increase in $n$ does not lead to substantial increase in accuracy of the solution, therefore the approximate solution (4.5) can be considered practically exact. This is also confirmed by the fact that the difference between values of function $\psi(r)$ in the form (4.4) and (4.5) does not exceed 2.5% for all $0 \leqslant r < 1$.

The value of the force $P$ acting on the die will also be determined from Eq.

$$P = a \int_{-1}^{1} \psi(r)\, dr \tag{4.6}$$

For cases (4.4) and (4.5) we obtain $P = 8.791\ \Delta\gamma a$ and $P = 8.794\ \Delta\gamma a$, respectively. In the paper [11] by a completely different method $P = 8.80\ \Delta\gamma a$ was obtained for the case under examination.

In this manner the concrete example given and other investigated examples, which are not presented for the sake of brevity, show that the convergence of the method presented in this paper is sufficiently high for intermediate values of parameter $\lambda$ ($\frac{1}{2} < \lambda < 2$). This method can serve as a reliable means for practical solution of integral Eq. (1.3) over the indicated range of variation in $\lambda$, providing for certain joining with asymptotic solutions of this equation for large and small $\lambda$.

**5. Reduction of basic integral equation of plane mixed problems to a solution of an infinite algebraic system.** Let us examine the integral equation

$$\int_{-1}^{1} \varphi(\xi)\, M\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x) \qquad (|x| \leqslant 1) \tag{5.1}$$

where the kernel $M(t)$ has the form (3.14) and function $L(u)$ satisfies as before conditions (1.2).

Without destroying generality we shall further assume that functions $\varphi(x)$ and $f(x)$ are even ("even" variant of integral Eq. (5.1)) because the solution for the "uneven" variant can be obtained by differentiation with respect to $x$ of the solution which was constructed for same even case [12] by a specific method.

Using integral [2]

$$\int_0^\infty \frac{\cos u - \cos ut}{u}\, du = \ln|t| \tag{5.2}$$

we rewrite integral Eq. (5.1) in the form

$$-\int_{-1}^{1} \varphi(\xi) \ln\frac{|\xi - x|}{\lambda}\, d\xi = \pi f(x) + \int_{-1}^{1} \varphi(\xi)\, F\left(\frac{\xi - x}{\lambda}\right) d\xi \qquad (|x| \leqslant 1) \tag{5.3}$$

$$F(t)=\int_0^\infty \frac{[1-L(u)]\cos ut-\cos u}{u}\,du$$

Based on properties (1.2) of function $L(u)$ it is not difficult to show that function $F(t)$ is continuous together with all its derivatives for all $-\infty < t < \infty$.

We shall seek solution $\varphi(\xi)$ of Eq. (5.3) in class $L$ ( $-(1,1)$. Then for $\lambda \to \infty$ the integral Eq. (5.3) degenerates into the following:

$$\int_{-1}^{1} \varphi_0(\xi)\left[-\ln\frac{|\xi-x|}{\lambda}-F(0)\right]d\xi=\pi f(x) \qquad (|x|\leqslant 1) \tag{5.4}$$

It is known that the contact problem for an elastic half-plane is reduced to such an equation. Solution of this integral equation in the form of singular integrals is most common. In our view the solution not containing singular integrals is more convenient for practical application. This solution was first found in paper [13].

Without dwelling on details we note that such a solution can be obtained quite easily by the method outlined in Section 1 for the determination of integral Eq. (1.9) and has the form

$$\varphi_0(x)=\frac{P}{\pi\sqrt{1-x^2}}-\frac{1}{\pi}\frac{d}{dx}x\int_x^1\frac{dt}{\sqrt{t^2-x^2}}\frac{d}{dt}\int_{-t}^{t}\frac{f(\xi)\,d\xi}{\sqrt{t^2-\xi^2}}$$

$$P=\int_{-1}^{1}\varphi_0(\xi)\,d\xi=\frac{1}{\ln 2\lambda-F(0)}\int_{-1}^{1}\frac{f(\xi)\,d\xi}{\sqrt{1-\xi^2}} \tag{5.5}$$

We can show, but we shall not dwell on it in detail, that solution (5.5) of the even variant of the integral Eq. (5.4) has a meaning for at least $f(x)\in H^{\alpha}(-1,1)$ and $\alpha>\frac{1}{2}$.

Further we shall assume that $f''(x)$ is bounded for $x\in[-1,1]$. Then the first Eq. (5.5) can be rewritten in the form

$$\varphi_0(x)=\frac{1}{\pi\sqrt{1-x^2}}\left[P+\int_{-1}^{1}\frac{f'(\xi)\,\xi\,d\xi}{\sqrt{1-\xi^2}}\right]-\frac{1}{\pi}\int_x^1\frac{dt}{t\sqrt{t^2-x^2}}\int_{-t}^{t}\frac{\xi\,[f'(\xi)\,\xi]'\,d\xi}{\sqrt{t^2-\xi^2}} \tag{5.6}$$

The solution of integral Eq. (5.4) for the uneven variant can be obtained very easily from (5.6) as indicated above (*).

*) We note that the solution of integral Eq. (5.4) for the uneven variant can also be found by a method different from the one presented in [12]. In fact, if both sides of Eq. (5.4) are differentiated with respect to $x$ and then if in the obtained relationship it is taken into account that functions $\varphi(0)(x)$ and $f(x)$ are even or uneven, this relationship can be presented in the respective form

$$\int_{-1}^{1}\frac{\varphi_0(\xi)}{\xi^2-x^2}\,d\xi=\pi f'(x)\,x^{-1},\qquad \int_{-1}^{1}\frac{\xi\psi_0(\xi)}{\xi^2-x^2}\,d\xi=\pi g'(x)\qquad(|x|\leqslant 1) \tag{i}$$

From this it is seen that

$$\varphi_0(x)=x\psi_0(x),\qquad f'(x)=xg'(x)\qquad(|x|\leqslant 1),\qquad P=M=\int_{-1}^{1}\xi\psi_0(\xi)\,d\xi \tag{ii}$$

In this manner solution of integral Eq. (5.4) for the uneven variant is obtained if in Eq. (5.6) which satisfies first Eq. of (*i*) substitutions are performed according to (*ii*). To determine the value of $M$ both sides of second Eq. of (*i*) are multiplied by $\sqrt{1-x^2}dx$ and we integrate from $-1$ to $+1$, then rearranging integrals in the left part of the obtained relationship and taking the inner integral, we find

$$M=-\int_{-1}^{1}g'(x)\sqrt{1-x^2}\,dx \tag{iii}$$

On the basis of Eq. (5.6) the following theorem can be proved in a manner which is analogous to the procedure in Section 1.

Theorem 4. If $f''(x)$ is bounded then $\varphi_0(x)$ has the form

$$\varphi_0(x) = \omega(x)(1-x^2)^{-1/2} \tag{5.7}$$

where $\omega(x) \in H^{1/2}(-1, 1)$, i.e. $\varphi_0(x) \in L(-1, 1)$.

The following more general theorem also applies.

Theorem 5. If function $f(x)$ is such that its $(n+2)$ derivative is bounded for $x \in [-1, 1]$, then function $\varphi_0(x)$ has the form (5.7) where $\omega^n(x) \in H^{1/2}(-1, 1)$.

We shall now seek the solution of integral Eq. (5.3) from the class $L(-1, 1)$ in the form

$$\varphi(x) = \varphi_0(x) + \varphi_1(x) \tag{5.8}$$

where $\varphi_0(x)$ is the solution of integral Eq. (5.4). For the correction function $\varphi_1(x)$ we obtain the integral Eq.

$$-\int_{-1}^{1} \varphi_1(\xi) \ln\frac{|\xi - x|}{\lambda} d\xi = \pi f_*(x) + \int_{-1}^{1} \varphi_1(\xi) F\left(\frac{\xi - x}{\lambda}\right) d\xi \qquad (|x| \leqslant 1)$$

$$f_*(x) = \frac{1}{\pi}\int_{-1}^{1} \varphi_0(\xi)\left[F\left(\frac{\xi - x}{\lambda}\right) - F(0)\right] d\xi \tag{5.9}$$

We note that by virtue of properties of function $F(t)$ pointed out above and the condition $\varphi(x) \in L(-1, 1)$ or by taking into account of Theorem 4, $\varphi_1(x) \in L(-1, 1)$, the entire right side of the integral Eq. (5.9), as function of $x \in [-1, 1]$, is continuous with all derivatives.

Then on the basis of Theorem 5 we can conclude that the general solution of integral Eq. (5.9), if it exists in $L(-1, 1)$, for any value $\lambda \in (0, \infty)$ has the form

$$\varphi_1(x) = \Omega(x)(1-x^2)^{-1/2} \tag{5.10}$$

where $\Omega(x)$ with all derivatives is a continuous function for $x \in [-1, 1]$.

Now let us represent the function $F(t)$ of the type (5.3) in the form of the following double series in Chebyshev polynomials:

$$F\left(\frac{\xi - x}{\lambda}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}(\lambda) T_i(x) T_j(\xi) \tag{5.11}$$

Functions $f_*(x)$ and $\Omega(x)$ entering into Eqs. (5.9) and (5.10) are also expanded in series

$$f_*(x) = \sum_{i=0}^{\infty} R_{i*} T_{2i}(x), \qquad \Omega(x) = \sum_{i=0}^{\infty} S_i T_{2i}(x) \tag{5.12}$$

By virtue of above noted properties of functions $F(t)$, $f_*(x)$ and $\Omega(x)$, series (5.11) and (5.12) converge uniformly, respectively to $F(t)$ for all variables $(x, \xi) \in [-1, 1]$ and any arbitrary value of parameter $\lambda \in (0, \infty)$, to $f_*(x)$ and $\Omega(x)$ for all values in the interval $-1 \leqslant x \leqslant 1$.

Utilizing known orthogonality property of Chebyshev [2] polynomials, we obtain for coefficients $c_{ij}(\lambda)$ of the series (5.11) Expression

$$c_{mn}(\lambda) = \frac{1}{\pi^2}\beta_{mn}\int_{-1}^{1}\int_{-1}^{1} F\left(\frac{\xi - x}{\lambda}\right)\frac{T_m(x) T_n(\xi)\, dx\, d\xi}{\sqrt{(1-x^2)(1-\xi^2)}} \tag{5.13}$$

$$(\beta_{00} = 1,\ \beta_{m0} = \beta_{0n} = 2,\ \beta_{mn} = 4)$$

We note that in the following only values $c_{2m,\ 2n}(\lambda)$ will be needed, because according to assumption, functions $\varphi(x)$ and $f(x)$, and therefore also functions $\varphi_1(x)$ and $f_*(x)$, are even.

Substituting into (5.13) Expression $F(t)$ in the form (5.3) and utilizing integral [2]

$$\int_{-1}^{1} \frac{T_{2i}(x)\cos ax\,dx}{\sqrt{1-x^2}} = (-1)^i \pi J_{2i}(a) \tag{5.14}$$

we obtain another equation for $c_{2m,\,2n}(\lambda)$

$$c_{00} = \int_0^\infty \frac{[1-L(u)]\,J_0^2(u/\lambda) - \cos u}{u}\,du \tag{5.15}$$

$$c_{2m,\,2n} = \beta_{2m,\,2n}(-1)^{m+n}\int_0^\infty [1-L(u)]\,J_{2m}\left(\frac{u}{\lambda}\right)J_{2n}\left(\frac{u}{\lambda}\right)\frac{du}{u} \qquad (m+n>0)$$

Let us proceed to determination of coefficients $R_{i*}$. Utilizing the second Eq. (5.9) and (5.11) we obtain (5.16)

$$R_{i*} = \sum_{n=0}^\infty c_{2i,\,2n}(\lambda)\,\Psi_n, \quad \Psi_n = \frac{1}{\pi}\int_{-1}^{1}\varphi_0(\xi)\,T_{2n}(\xi)\,d\xi, \quad R_{0*} = \sum_{n=0}^\infty c_{0,\,2n}(\lambda)\Psi_n - F(0)\Psi_0$$

For evaluation of integrals $\Psi_n$ we multiply both parts of the integral Eq. (5.4) term by term by $(1-x^2)^{-1/2}\,T_{2n}(x)\,dx$ and integrate with respect to $x$ from $-1$ to 1. Then rearranging integrals in the left part of the obtained relationship and taking advantage of Eq. (2.7) of the paper [1] we find

$$\Psi_0 = R_0\,[\ln 2\lambda - F(0)]^{-1}, \qquad \Psi_n = nR_n \tag{5.17}$$

$$\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)\,T_{2n}(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 2R_0 & \text{for } n=0 \\ R_n & \text{for } n>0 \end{cases}$$

We note that quantities $R_n$ are coefficients of expansion of function $f(x)$ in series with respect to Chebyshev polynomials of the form (5.12).

Finally, we obtain a relationship for determination of coefficients $S_i$ in the second Eqs. (5.12). Substituting into integral Eq. (5.9) functions $\varphi_1(\xi)$, $f_*(x)$ and $F(t)$ in the form (5.10) to (5.12) and evaluating integrals (it is necessary to take advantage of Eq. (2.7) of the paper [1] and of the known orthogonality property of Chebyshev polynomials), we obtain a relationship the left and right parts of which contain series in Chebyshev polynomials. Equating coefficients of both parts for polynomials of the same index we obtain an infinite system of linear algebraic equations of the form (3.1) for determination of $S_i$, where

$$x_0 = S_0 \ln 2\lambda, \qquad x_i = S_i(2i)^{-1}, \qquad a_{i0} = c_{2i,\,0}(\ln 2\lambda)^{-1}, \qquad a_{ik} = kc_{2i,\,2k} \tag{5.18}$$

For large values of parameter $\lambda$ we can obtain an asymptotic representation of the form (3.3) for coefficients of the infinite system $a_{ik}$ in a manner analogous to what was done in Section 3. If in addition it is assumed in this representation that $i$ and $k$ are large, we shall have

$$a_{ik} = O\left(\frac{p^{2p+1/2}}{(2\lambda\nu)^{2p}\,i^{2i+1/2}k^{2k-1/2}}\right) \qquad (p=i+k) \tag{5.19}$$

In analogy to what was presented in Section 3, asymptotically for small $\lambda$ we can obtain $a_{ik} = 0$ $(i \neq k)$ and $a_{ii} = 1$.

Now with respect to infinite system (3.1), (5.18), the same conclusions as in Section 3 may be drawn. Namely, for all $\lambda > \nu^{-1}$ the system will be quasi-completely regular. Its free terms $R_{i*}$ are bounded from above and for $i \to \infty$ they approach zero by virtue of uniform convergence of the first series (5.12). For small values of parameter $\lambda$ the infinite system becomes unstable.

For finding of approximate solutions of infinite system (3.1), (5.18) it is convenient to take advantage of the reduction method. The reduced (truncated) system has the form (3.11), its coefficients, as can be easily noted, form an almost triangular matrix. This makes it considerably easier to obtain concrete results.

With decreasing parameter $\lambda$ convergence of series (5.11) on the line $\xi = x$ becomes poorer and in order to preserve given accuracy of the approximate solution, the number of equations $n+1$ in the reduced (truncated) system must be increased. Consequently, good convergence of the method presented in this Section for solution of integral Eq. (5.1) must be expected only for large and intermediate values of parameter $\lambda$.

As in Section 3 we shall briefly dwell on other methods of approximate solution of integral Eq. (5.5).

For large values of parameter $\lambda$ it is possible to obtain an asymptotic solution determined by Eqs. (2.9) and (2.10) in the paper [14]. As a rule it can be used for $2 \leqslant \lambda < \infty$.

For construction of the asymptotic for the solution of the integral Eq. (5.1) for small $\lambda$ we represent the integral equation in the form of a system of three integral equations which are equivalent to it(*)

$$\int_{-1}^{\infty} \beta\left(\frac{1+\xi}{\lambda}\right) M\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f(x) + \int_{-\infty}^{-1}\left[\beta\left(\frac{1-\xi}{\lambda}\right) - v(\xi)\right] M\left(\frac{\xi-x}{\lambda}\right) d\xi \quad (-1 \leqslant x < \infty)$$

$$\int_{-\infty}^{1} \beta\left(\frac{1-\xi}{\lambda}\right) M\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f(x) + \int_{1}^{\infty}\left[\beta\left(\frac{1+\xi}{\lambda}\right) - v(\xi)\right] M\left(\frac{\xi-x}{\lambda}\right) d\xi \quad (-\infty < x \leqslant 1) \tag{5.20}$$

$$\int_{-\infty}^{\infty} v(\xi)\, M\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f(x) \qquad (-\infty < x < \infty)$$

under the condition

$$\varphi(\xi) = \beta\left(\frac{1+\xi}{\lambda}\right) + \beta\left(\frac{1-\xi}{\lambda}\right) - v(\xi) \tag{5.21}$$

where the function $f(x)$ is continued in an arbitrary manner in the region $-\infty < x \leqslant -1$ and $1 \leqslant x < \infty$ with preservation of sufficient smoothness.

The solution of the last integral Eq. (5.20) can be obtained easily through application of the theorem on convolution for a Fourier transform.

The first two integral Eqs. (5.20) are reduced to one through a change of variables

$$\int_{0}^{\infty} \beta(\tau) M(\tau - t)\, d\tau = \frac{\pi}{\lambda} f(1-\lambda t) +$$

$$+ \int_{2/\lambda}^{\infty} [\beta(\tau) - v(1-\lambda\tau)]\, M\left(\frac{2}{\lambda} - \tau - t\right) d\tau \quad (0 \leqslant t < \infty) \tag{5.22}$$

The asymptotic solution for small $\lambda$ of the integral Eq. (5.22) can be found by the method of successive approximations. Here, at each step it is necessary to find the solution for one and the same integral equation of Wiener-Hopf, but with different right-hand sides.

The zeroth approximation corresponding to the zeroth term of the asymptotic for the solution of the integral Eq. (5.1) for small $\lambda$ is found from Eq.

$$\int_{0}^{\infty} \beta_0(\tau) M(\tau - t)\, d\tau = \frac{\pi}{\lambda} f(1-\lambda t) \qquad (0 \leqslant t < \infty) \tag{5.23}$$

Examination of concrete problems shows [16 and 17] that, as a rule, the zeroth term of the asymptotic of the solution for small $\lambda$ reliably matches with the asymptotic solution for large $\lambda$ [14], thus providing a complete solution of the problem.

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*) In the papers [10 and 15] a different approach to the construction of the asymptotic of integral Eq. (5.1) is presented for small $\lambda$ under the assumption that function $L(u)$ entering into kernel $M(t)$ is meromorphic.

In those cases where matching is not achieved with required accuracy, we may use the approximate solution for small $\lambda$ constructed from Eq. (5.21) on the basis of first (or higher) approximation of solution of Eq. (5.22).

There is also another possibility, in our opinion even more convenient, i.e. to utilize the method of approximate solution of integral Eq. (5.1) presented in this Section, as a "connecting bridge" between the asymptotic solution for large $\lambda$ and the zeroth term of the asymptotic for small $\lambda$.

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